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HAMILTONIAN APPROACH TO 2+1 DIMENSIONAL GRAVITY

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Abstract

It is shown that the reduced particle dynamics of $2+1$ dimensional gravity in the maximally slicing gauge is of hamiltonian nature. We give the exact diffeomorphism which transforms the expression of the spinning cone geometry in the Deser, Jackiw, 't Hooft gauge to the maximally slicing gauge. It is explicitly shown that the boundary term in the action, written in hamiltonian form gives the Hamiltonian for the reduced particle dynamics. The quantum mechanical translation of the two particle Hamiltonian is given and the Green function computed.

1 Introduction

Gravity in 2+1 dimensions [1] has been the object of vast interest both at the classical and quantum level. Several approaches have been pursued [1, 2, 3, 4]. In [5, 6] the maximally slicing gauge, or instantaneous York gauge, was introduced. The application of such a gauge is practically restricted to open universes with the topology of the plane; here however it will prove a very powerful tool.

The approach developed in [5, 6] is first order. In [7, 8] the same gauge was exploited in the second order ADM approach; this approach turns out to be more straightforward than the previous one and being strictly canonical lends itself to be translated at the quantum level. After giving a concise summary of the results of the previous papers [7, 8], we derive in sect.3 generalized conservation laws starting from the time evolution of the analytic component of the energy momentum tensor of the Liouville theory which underlies the conformal factor describing the space metric. In sect.4 we prove explicitly the hamiltonian nature of the reduced particle dynamics i.e. the fact that one can give a hamiltonian description of the time development of the system in terms of the position and momenta of the particles.

While for the two particle case the result is elementary, for three particles it involves the exploitation of the Garnier equations, related to the isomonodromic transformations of a fuchsian problem. For more than three particles the proof of the hamiltonian nature of the reduced equations of motion relies on a conjecture

by Polyakov [9] on the relation between the regularized Liouville action and the accessory parameters of the $SU(1,1)$ Riemann-Hilbert problem. Such a conjecture has been proved by Zograf and Takhtajan [9] for the special cases of parabolic singularities and elliptic singularities of finite order, but up to now a proof for general elliptic singularities is absent.

In sect. 5 we give the exact diffeomorphism which relates the conical metric of Deser, Jackiw and 't Hooft (DJH) in the presence of angular momentum to its description in the maximally slicing gauge.

In sect.6 we connect the results of sect.4 with the boundary terms of the gravitational action; it is explicitly shown that the dynamics is described completely by such boundary terms of the action.

In the last section it is shown that quantum Hamiltonian for the two body problem is the logarithm of the Laplace-Beltrami operator on a cone whose aperture is given by the total energy of the system and is independent of the masses of the two particles. This provides a complete proof of the conjecture of 't Hooft [10] about the two particle dynamics.

The quantum mechanical problem with more than two particles requires a more explicit knowledge of the Hamiltonian which is related to the auxiliary parameters β_B. The existence of those parameters is assured by the solvability of the Riemann-Hilbert problem and one can try to produce a perturbative expansion of them in some limit situations. Here however the ordering problem is likely to be more acute.

2 Hamiltonian approach

To make the paper relatively self-contained we shall summarize in this section some results of the papers [7, 8]. With the usual ADM notation for the metric [11]

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{1}$$

the gravitational action expressed in terms of the canonical variables is [12]

$$\begin{aligned} S_{Grav} &= \int dt \int_{\Sigma_t} d^D x \left[\pi^{ij}\dot{g}_{ij} - N^i H_i - NH\right] - 2\int dt \int_{B_t} d^{(D-1)}x\, r_\alpha \pi^{\alpha\beta}_{(\sigma_{Bt})} N_\beta \\ &+2\int dt \int_{B_t} d^{(D-1)}x\,\sqrt{\sigma_{Bt}}N\left(K_{B_t} + \frac{\eta}{\cosh\eta}\mathcal{D}_\alpha v^\alpha\right) \end{aligned} \tag{2}$$

where for the detailed meaning of the symbols and the explicit form of H and H_i and of the matter action we refer to [7]. In the $K = 0$ gauge and using the complex coordinates $z = x + iy$ the diffeomorphism constraint is simply solved by

$$\pi^{\bar{z}}{}_z = -\frac{1}{2\pi}\sum_n \frac{P_n}{z - z_n} \tag{3}$$

subject to the restriction $\sum_n P_n = 0$ [8]. Always for $K = 0$ and using the conformal gauge for the space metric i.e. $g_{ij}dx^i dx^j = e^{2\sigma}dzd\bar{z}$ the hamiltonian constraint takes the form of the following inhomogeneous Liouville equation

$$2\Delta\tilde{\sigma} = -e^{-2\tilde{\sigma}} - 4\pi\sum_n \delta^2(z - z_n)(\mu_n - 1) - 4\pi\sum_A \delta^2(z - z_A) \tag{4}$$

where $e^{2\sigma} = 2\pi^{\bar{z}}{}_z\pi^z{}_{\bar{z}}e^{2\tilde{\sigma}}$, μ_n are the particle masses divided by 4π, z_n the particle positions and z_A the positions of the $(\mathcal{N} - 2)$ apparent singularities i.e. of the zeros of eq.(3). The Lagrange multipliers N and N^z where expressed in terms of $\tilde{\sigma}$ through

$$N = \frac{\partial(-2\tilde{\sigma})}{\partial M}; \qquad N^z = -\frac{2}{\pi^{\bar{z}}{}_z(z)}\partial_z N + g(z), \tag{5}$$

with

$$g(z) = \sum_B \frac{\partial\beta_B}{\partial M}\frac{1}{z - z_B}\frac{\mathcal{P}(z_B)}{\prod_{C\neq B}(z_B - z_C)} + p_1(z) \tag{6}$$

and $\mathcal{P}$ is defined by

$$-\frac{\pi^{\bar z}{}_z(z)}{2} = \frac{1}{4\pi}\sum_n \frac{P_{nz}}{z - z_n} \equiv \frac{\prod_B (z - z_B)}{\mathcal{P}(z)}. \tag{7}$$

$p_1(z) = c_0(t) + c_1(t)z$ is a first order polynomial. The role of the first term in $g(z)$ is to cancel the poles arising in the first term of eq.(5) due to the zeros of $\pi^{\bar z}{}_z$ and β_B are the accessory parameters of the fuchsian differential equation [13] which underlies the solution of the Liouville equation (4). The equations for the particle motion are given by [7]

$$\dot z_n = -N^z(z_n) = -g(z_n); \qquad \dot P_{nz} = P_{na}\frac{\partial N^a}{\partial z} - m_n\frac{\partial N}{\partial z}. \tag{8}$$

If we want a reference frame which does not rotate at infinity the linear term in $p_1(z)$ must be chosen so as to cancel in N^z the term increasing linearly at infinity; such a choice is unique and given by $-z/(\sum_n P_n z_n)$.

In the simple two particle case the equations of motion in the relative coordinates $z'_2 = z_2 - z_1$, $P' = P_2 = -P_1$ are $\dot z'_2 = 1/P'_z$ and $\dot P'_z = -\mu/z'_2$ whose solution

$$z'_2 = \text{const}\ [(1-\mu)(t - t_0) - iL/2]^{\frac{1}{1-\mu}} \tag{9}$$

agrees with the solution found in [5].

3 Virasoro generators and conservation laws

In ref.[7] the following equation was derived from the ADM formalism, with regard to the time evolution of the function $Q(z)$ appearing in the fuchsian differential equation underlying $\sigma(z, \bar z)$

$$\dot Q(z) = \frac{1}{2}g'''(z) + 2g'(z)Q(z) + g(z)Q'(z). \tag{10}$$

$Q(z)$ can be understood as the analytic component of the energy momentum tensor of the Liouville theory governing the conformal factor $\tilde\sigma$. We convert eq.(10) into equations for the Laurent series coefficients L_n of $Q(z)$ obtaining the following equation of motion

$$\dot L_{-1} = \frac{c_1}{2}(\sum_n \beta_n + \sum_B \beta_B); \dot L_0 = -\frac{c_0}{2}(\sum_n \beta_n + \sum_B \beta_B); \dot L_1 = -c_0 L_0 - c_1 L_1. \tag{11}$$

The equation for L_{-1} is a simple consistency requirement on the first Fuchs relation $\sum_n \beta_n + \sum_B \beta_B = 0$. Equation for $\dot L_0$ tells us that the value of L_0, provided by the second Fuchs relation,

$$4L_0 = 2\sum_n \beta_n z_n + 2\sum_B \beta_B z_B + \sum_n (1 - \mu_n^2) - 3(\mathcal{N} - 2) = 1 - \mu_\infty^2 = 1 - (1-\mu)^2. \tag{12}$$

is constant in time, namely the total mass of the universe is constant. Combining the derivative of (12) with respect to μ with the equations of motion provides the generalized conservation law, obviously related to the dilatations,

$$\frac{d}{dt}(\sum_n P_n z_n) = 1 - \mu. \tag{13}$$

In the two particle case eq.(13) is simply $P'z'_2 = (1-\mu)(t-t_0)-iL$. As can be easily checked the Hamiltonian in this case is given by the sum of two conserved Hamiltonians i.e. $H = h + \bar{h}$ with $h = \ln(P'z'^{\mu}_2)$. Taking the ratio of $P'z'^{\mu}_2 = \exp(h) = \text{const.}$ with the previous equation we obtain the solution eq.(9) without the need to solve the system (8). Also the equation for L_1 can be solved; in the non rotating frame and for $c_0(t) = 0$ it gives

$$\sum_n P_n z_n^2 = (k_2 t + k_3)[(1-\mu)t - ib]^{\frac{1}{1-\mu}} \tag{14}$$

where k_2, k_3 are constants.

4 Hamiltonian nature of the reduced dynamics

Starting form the ADM action we have reached the particle equations of motion following a canonical procedure; thus, we expect equations (8) to be derivable from a Hamiltonian. The present section is devoted to the direct proof that such equations are indeed generated by a Hamiltonian and to the construction of such Hamiltonian. Since $\sum_n P_n = 0$ it is useful to perform a canonical transformation to the variables $z'_n = z_n - z_1, P'_n = P_n, n = 2\dots\mathcal{N}$. Using the definition of $\mathcal{P}$ and the defining property of the locations of the apparent singularities z_B, $\pi^{\bar{z}}_z(z_B) = 0$, one finds

$$\dot{z}'_n = -\sum_B \frac{\partial\beta_B}{\partial\mu}\frac{\partial z'_B}{\partial P'_n} - c_1(t)z'_n \quad \text{and} \quad \dot{P}'_n = \frac{\partial\beta_n}{\partial\mu} + \sum_B \frac{\partial\beta_B}{\partial\mu}\frac{\partial z'_B}{\partial z'_n} + c_1(t)P'_n. \tag{15}$$

One can get rid of $c_1(t)$ in eq.(15) by means of a canonical transformation. In the three body case one finds that the Hamiltonian is of the form

$$H(z'_2, z'_3, P'_2, P'_3) = -\int_{z_0}^{z'_A} \frac{\partial\beta_A}{\partial\mu}(z'_2, z'_3, z''_A)\, dz''_A + f(z'_2, z'_3) \tag{16}$$

where z'_A is a function of z'_n and P'_n through the relation $\pi^{\bar{z}}{}_z(z_B) = 0$. Such Hamiltonian provides the equations of motion for $\dot{z}'_n$ and by use of the Garnier equations one proves the existence at the local level of the function $f(z'_2, z'_3)$ which provides also the equations for $\dot{P}'_n$. The natural extension of (16) to $\mathcal{N}$ particles is

$$H(z'_2, .., z'_{\mathcal{N}}, P'_2, .., P_{\mathcal{N}}) = -\int_{\{z_0\}}^{\{z'_B\}} \sum_B \frac{\partial\beta_B}{\partial\mu}(z'_2, .., z'_{\mathcal{N}}, z''_A, .., \mu) dz''_B + f(z'_2, .., z'_{\mathcal{N}}). \tag{17}$$

In order eq. (17) to make sense we need the form $\omega = \sum_A \partial\beta_A/\partial\mu \;\; dz_A$ to be exact. Such a property is a consequence of a conjecture due to Polyakov [9] which states

$$\sum_n \beta_n dz_n + \sum_B \beta_B dz_B = -\frac{1}{2\pi} dS_\epsilon \tag{18}$$

being S_ϵ regularized Liouville action [14]

$$S_\epsilon[\phi] = \frac{i}{2}\int_{X_\epsilon} (\partial_z\phi\partial_{\bar{z}}\phi + \frac{e^\phi}{2}) dz \wedge d\bar{z} - \frac{i}{2}\sum_n (1-\mu_n)\oint_n \phi(\frac{d\bar{z}}{\bar{z}-\bar{z}_n} - \frac{dz}{z-z_n})$$
$$+\frac{i}{2}\sum_B \oint_B \phi(\frac{d\bar{z}}{\bar{z}-\bar{z}_B} - \frac{dz}{z-z_B}) - \frac{i}{2}(\mu-2)\oint_\infty \phi(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z})$$

$$-\pi(\sum_n(1-\mu_n)^2-(\mathcal{N}-2)-(\mu-2)^2)\ln\epsilon^2. \tag{19}$$

Such a conjecture has been proved by Zograf and Tahktajan [9] for fuchsian differential equations with parabolic singularities; in addition they remark that the proof can be extended in a straightforward way to the case of elliptic singularities of finite order. The extension of the proof to the general elliptic case is still missing.

Is is now straightforward to prove that

$$H=\frac{1}{2\pi}\frac{\partial S_\epsilon}{\partial\mu} \tag{20}$$

is the correct Hamiltonian as

$$-\frac{\partial H}{\partial z'_n}=-\frac{1}{2\pi}\frac{\partial^2 S_\epsilon}{\partial\mu\partial z'_n}-\frac{1}{2\pi}\sum_B\frac{\partial^2 S_\epsilon}{\partial\mu\partial z'_B}\frac{\partial z'_B}{\partial z'_n}=\frac{\partial\beta_n}{\partial\mu}+\sum_B\frac{\partial\beta_B}{\partial\mu}\frac{\partial z'_B}{\partial z'_n}. \tag{21}$$

The complete form in the non rotating frame is given by

$$H=\ln\left[(\sum_n P_n z_n)(\sum_n \bar P_n\bar z_n)\right]+\frac{1}{2\pi}\frac{\partial S_\epsilon}{\partial\mu}. \tag{22}$$

Note that this Hamiltonian, being time-independent, provides a further conservation law in the $\mathcal{N}-$particle problem. For a more complete discussion of the relation between the Polyakov conjecture and this Hamiltonian we refer the reader to [15].

5 The asymptotic metric

In order to understand the boundary terms we investigate the diffeomorphism which connects the metric for a spinning particle in the DJH gauge

$$ds^2=-(dT+Jd\phi)^2+dR^2+\alpha^2R^2d\phi^2=-(dT+Jd\phi)^2+\alpha^2r_0^2(\zeta^{\alpha-1})^2(d\zeta^2+\zeta^2d\phi^2), \tag{23}$$

with the metric of the same geometry in the $K=0$ gauge; here $R=r_0\zeta^\alpha$.

The metric (23) possesses two Killing vector fields, $\partial/\partial T$ and $\partial/\partial\phi$. A solution of the Einstein equations which complies to the York instantaneous gauge and with such Killing vectors is provided by

$$\pi^{\bar z}{}_z=-\frac{1}{2\pi z^2}\sum_n P_n z_n\equiv\frac{p(t)}{z},\quad e^{2\sigma}=\frac{|p(t)|^2}{(z\bar z)n^2}\frac{16\alpha^2}{\Lambda(t)^2}\frac{(\frac{z\bar z}{\Lambda(t)^2})^{-\alpha-1}}{(1-(\frac{z\bar z}{\Lambda(t)^2})^{-\alpha})^2} \tag{24}$$

and

$$N=\frac{1}{2\pi\alpha}\left[\frac{1}{2}\ln\left(\frac{k^2(t)r^{2\alpha}}{\Lambda(t)^{2\alpha}}\right)-1+\frac{1}{r^{2\alpha}/\Lambda(t)^{2\alpha}-1}\ln\left(\frac{k^2(t)r^{2\alpha}}{\Lambda(t)^{2\alpha}}\right)\right] \tag{25}$$

$$N^z=\frac{z}{\pi p(t)}\left[\frac{r^{2\alpha}/\Lambda(t)^{2\alpha}}{(r^{2\alpha}/\Lambda(t)^{2\alpha}-1)^2}\ln\left(\frac{k^2(t)r^{2\alpha}}{\Lambda(t)^{2\alpha}}\right)-\frac{1}{(r^{2\alpha}/\Lambda(t)^{2\alpha}-1)}\right] \tag{26}$$

where $\alpha=1-\mu=1-\frac{M}{4\pi}$ and $r=|z|$. While the dependence of $k(t)$ on time is arbitrary, the canonical equations for $\pi^z{}_{\bar z}$ and σ fix the time evolution of $p(t)$ and $\Lambda(t)$ to be

$$p(t)=-\frac{1}{2\pi}[\alpha t-ib],\quad \Lambda^2(t)=c_\Lambda[p(t)\bar p(t)]^{\frac{1}{\alpha}}. \tag{27}$$

This, in turn, implies that the asymptotic expansion $e^{2\sigma} \approx s^2(z\bar{z})^{-\mu}$ has to be time independent, in agreement with the fact that $\ln s^2$ will coincide with the Hamiltonian. As $\partial/\partial\theta$ is mapped into $\partial/\partial\phi$ the transformation must have the form

$$R = R(\rho, t); \qquad T = T(\rho, t); \qquad \phi = \theta + \omega(\rho, t). \tag{28}$$

Comparing the independent components of the metric, we are led to a set of differential equations which determines the unknown functions in eq. (28) (see [15] for details). At the end one finds

$$R^2 = \frac{1}{\alpha^2}\left[J^2 + \frac{r^{2\alpha}}{4c_\Lambda(t)\alpha^2}\left(1 - c_\Lambda|p(t)|^2 r^{-2\alpha}\right)^2\right] \tag{29}$$

$$\phi = \theta + \omega \equiv \theta + \frac{1}{\alpha}\arctan\left[2\pi\frac{c_\Lambda^{-1}r^{2\alpha} - |p(t)|^2 + 2\alpha^2 J^2}{2\alpha^2 Jt}\right] \tag{30}$$

$$T = \frac{t}{4\pi}\left[\ln\frac{r^2}{c_\Lambda} - \frac{1}{\alpha}\ln|p(t)|^2\right] - J\omega + h(t) \tag{31}$$

with $h(t)$ obeying the relation $\dot{h}(t) = 1/4\pi\alpha[\ln k^2(t) - 2\alpha^2 J^2/|p(t)|^2]$. This shows that two asymptotic solutions with different $k^2(t)$ are diffeomorphic to the same DJH metric and thus are diffeomorphic to each other. This explains why for any choice of $k^2(t)$ eqs.(24,25,26) are solutions of Einstein's equations. For large r eqs.(29,30,31) become

$$R^2 \approx \frac{r^{2\alpha}}{4c_\Lambda\alpha^4}, \quad \phi \approx \theta + \frac{\pi}{2\alpha}, \quad T \approx \frac{t}{4\pi}\ln\left(\frac{r^2}{c_\Lambda|p(t)|^{\frac{2}{\alpha}}}\right) - \frac{\pi J}{2\alpha} + h(t). \tag{32}$$

In the DJH gauge a finite transformation along the Killing vector $\partial/\partial T$ is simply given by $T \to T + c$ while in the York instantaneous gauge it is more complicated. The time-like Killing vector in the instantaneous York gauge is simply computed and given by

$$\frac{(2\pi)^3\alpha(\rho^2+1)|p(t)|^2}{\mathcal{D}}\frac{\partial}{\partial t} + \frac{8\pi^2 J\alpha^2}{\mathcal{D}}\frac{\partial}{\partial\theta} + \frac{4\pi\alpha^2 rt}{\mathcal{D}}\frac{\partial}{\partial r} \tag{33}$$

with

$$\mathcal{D} = 4\pi^2|p(t)|^2(\rho^2+1)[\ln\rho + 2\pi\alpha\dot{h}(t)] + \alpha^2 t^2(1-\rho^2) + 8\pi^2 J^2\alpha^2. \tag{34}$$

For large r the vector (33) reduces to

$$\frac{4\pi}{\ln\left(\frac{r^2}{c_\Lambda|p(t)|^{\frac{2}{\alpha}}}\right)}\left(\frac{\partial}{\partial t} + \frac{\alpha J c_\Lambda^\alpha}{\pi r^{2\alpha}}\frac{\partial}{\partial\theta} + \frac{\alpha t c_\Lambda^\alpha}{2\pi^2 r^{2\alpha}} r\frac{\partial}{\partial r}\right). \tag{35}$$

6 The Hamiltonian as a boundary term

In the $K = 0$ conformal gauge we have $\pi^{ij}\dot{g}_{ij} \equiv 0$ and thus the action of the particles plus gravity reduces to

$$S = \int dt\left(\sum_n P_{ni}\,\dot{q}_n^i - H_B\right) \tag{36}$$

where H_B is read from eq.(2). We want now to extract from H_B the reduced particle Hamiltonian and compare it to the Hamiltonian H derived directly from the particle equations of motion. It is not difficult to show that at large distances the only surviving boundary term is

$$H_B = -2\int_{B_t} d^{(D-1)}x\,\sqrt{\sigma_{Bt}}\,N K_{B_t}. \tag{37}$$

By inserting the metric eqs.(24,25,26) into the expression for K_{Bt} and σ_{Bt} we obtain for the integral

$$H_B = -4\pi N r_0 \partial_r [\ln(r e^{\sigma})] \tag{38}$$

and thus for large r_0 the boundary term becomes

$$H_B = -r_0 \ln r_0^2 (\frac{1}{r_0} + \partial_r \sigma) = (\mu - 1) \ln r_0^2. \tag{39}$$

We recall now that the equations of motion are obtained from the action by keeping the values of the fields fixed at the boundary, or equivalently by keeping fixed the intrinsic metric of the boundary[16]. In our case the variations should be performed keeping fixed the fields N, N^a, and σ at the boundary. We shall perform the computation for the boundary given by a circle of radius r_0 for a very large value of r_0. If we change the positions of particle positions and momenta, Λ varies and in order to keep the value of σ fixed at the boundary we must vary μ as to satisfy the following equality

$$\ln\{(\sum_n P_n z_n)(\sum_n \bar{P}_n \bar{z}_n)\} - \mu \ln r_0^2 + (\mu - 1) \ln \Lambda^2 - \ln 16\pi^2 \equiv -\mu \ln r_0^2 + \ln s^2 = \text{const.} \tag{40}$$

Thus

$$0 = -\delta\mu \ln r_0^2 + \sum_n (\delta z_n \frac{\partial \ln s^2}{\partial z_n} + \delta P_n \frac{\partial \ln s^2}{\partial P_n} + \text{c.c.}) + \delta\mu \frac{\partial \ln s^2}{\partial \mu} \tag{41}$$

i.e. for large r_0

$$\delta\mu \approx \frac{1}{\ln r_0^2} \sum_n (\delta z_n \frac{\partial \ln s^2}{\partial z_n} + \delta P_n \frac{\partial \ln s^2}{\partial P_n} + \text{c.c.}). \tag{42}$$

Substituting into eq.(39) we have

$$\delta H_B = \sum_n (\delta z_n \frac{\partial \ln s^2}{\partial z_n} + \delta P_n \frac{\partial \ln s^2}{\partial P_n} + \text{c.c.}) \tag{43}$$

i.e. apart for a constant H_B equals $\ln s^2$

$$H_B = \ln s^2 + \text{const.} = \ln \left[(\sum_n P_n z_n)(\sum_n \bar{P}_n \bar{z}_n) \right] + (\mu - 1) \ln \Lambda^2 + \text{const.} \tag{44}$$

However the Hamiltonian H_B is not the energy as usually defined i.e. the value of the boundary term when (N, N^i) take the values of the asymptotic time-like Killing vector. In our case due to the choice of the $K = 0$ gauge which vastly simplifies the dynamics, the (N, N^i) differ from the timelike asymptotic Killing vector. The energy of a solution is easily obtained in the DJH gauge, where one checks from the metric eq.(23) that with $(N, N^i) = (1, 0, 0)$ i.e. the normalized Killing vector, one obtains for H_B the value $4\pi(\mu - 1)$ as expected.
Under a complex scaling $z' = \alpha z$, $\ln \Lambda^2$ behaves as follows

$$\ln \Lambda^2 (\alpha z_n, \frac{P_n}{\alpha}) = \ln \Lambda^2 (z_n, P_n) + \ln(\alpha \bar{\alpha}) \tag{45}$$

which provides the following Poisson bracket

$$[H, \sum_n P_n z_n] = [\sum_n P_n z_n, (\mu - 1) \ln \Lambda^2] = \mu - 1 \tag{46}$$

and thus we have reached a hamiltonian derivation of the generalized conservation law

$$\sum_n P_n z_n = (1-\mu)(t-t_0) - ib. \tag{47}$$

We want now to relate the result eq.(44) to the results of sect.4. Let us now consider the derivative of the regularized Liouville action with respect to μ; as we are varying around a stationary point the only contribution is provided by the terms in eq.(19) which depend explicitly on μ i.e.

$$\frac{\partial S_\epsilon}{\partial \mu} = -\frac{i}{2}\oint_\infty \phi(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}) - 2\pi(\mu-2)\ln\epsilon^2 \tag{48}$$

and as $\phi \equiv -2\tilde{\sigma} \approx \ln 8(1-\mu)^2 + (\mu-2)\ln z\bar{z} - (\mu-1)\ln\Lambda^2$ we have

$$\frac{\partial S_\epsilon}{\partial \mu} = -2\pi\ln 8(1-\mu)^2 + 2\pi(\mu-1)\ln\Lambda^2. \tag{49}$$

Thus we can rewrite eq.(44) as

$$H = \ln\left[(\sum_n P_n z_n)(\sum_n \bar{P}_n \bar{z}_n)\right] + \frac{1}{2\pi}\frac{\partial S_\epsilon}{\partial \mu} + \text{const} \tag{50}$$

in agreement with the result of sect.4 obtained through Polyakov's conjecture.

7 Quantization: the two particle case

Several quantization schemes for 2+1 dimensional gravity has been proposed [17].
The classical two particle Hamiltonian in the reference system which does not rotate at infinity given in sect.3 can be written as

$$H = \ln((x^2+y^2)^\mu((P_x)^2 + (P_y)^2)). \tag{51}$$

Keeping in mind that with our definitions P is the momentum multiplied by $16\pi G_N/c^3$, applying the correspondence principle we have

$$[x, P_x] = [y, P_y] = i l_P \tag{52}$$

where $l_P = 16\pi G_n \hbar/c^3$, all the other commutators equal to zero. H is converted into the operator

$$\ln[-(x^2+y^2)^\mu \Delta] + \text{ constant}. \tag{53}$$

i.e. the logarithm is the Laplace-Beltrami Δ_{LB} operator on the metric $ds^2 = (x^2+y^2)^{-\mu}(dx^2+dy^2)$. Following an argument similar to the one presented in [18] one easily proves that if we start from the domain of Δ_{LB} given by the infinitely differentiable functions of compact support C_0^∞ which can also include the origin, then Δ_{LB} has a unique self-adjoint extension and $\ln(\Delta_{LB})$ is also self-adjoint (see [15] for details).
Deser and Jackiw [19] have considered the quantum scattering of a test particle on a cone both at the relativistic and non relativistic level. Most of the techniques developed there can be transferred here: instead of the Hamiltonian $(x^2+y^2)^\mu(p_x^2+p_x^2)$ we have now the Hamiltonian $\ln[(x^2+y^2)^\mu(p_x^2+p_y^2)]$. The partial wave eigenvalue differential equation

$$(r^2)^\mu[-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{n^2}{r^2}]\phi_n(r) = k^2\phi_n(r) \tag{54}$$

with $\mu = 1-\alpha$ is solved by $\phi_n(r) = J_{\frac{|n|}{\alpha}}(\frac{k}{\alpha}r^\alpha)$. A straightforward calculation, using the spectral representation of the above operator, leads to the associated Green function, which can be exploited to analyze the scattering problem

$$G(z,z',t) = \frac{2}{\alpha\Gamma(\frac{ict}{l_P})rr'}\left(\frac{r^\alpha + {r'}^\alpha}{2\alpha}\right)^{2ict/l_P}$$
$$\sum_n \frac{e^{in(\phi-\phi')}}{2\pi}\frac{\Gamma(\frac{|n|}{\alpha}+1-\frac{ict}{l_P})}{\Gamma(\frac{|n|}{\alpha}+1)}\rho^{\frac{|n|}{\alpha}+1}{}_2F_1(\frac{|n|}{\alpha}+1-\frac{ict}{l_P};\frac{|n|}{\alpha}+\frac{1}{2};2\frac{|n|}{\alpha}+1;4\rho) \tag{55}$$

where $\rho = r^\alpha {r'}^\alpha/(r^\alpha + {r'}^\alpha)$.

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References

[1] A. Staruszkiewicz, Acta Phys. Polonica 24 (1963) 734; S. Deser, R. Jackiw and G. 't Hooft, Ann. Phys. (NY) 152 (1984) 220; S. Deser and R. Jackiw, Ann. Phys. 153 (1984) 405.

[2] G. 't Hooft, Class. Quantum Grav. 9 (1992) 1335; Class. Quantum Grav. 10 (1993) 1023.

[3] G. 't Hooft, Class. Quantum Grav. 10 (1993) 1653; Class. Quantum Grav. 13 (1996) 1023.

[4] H. Waelbroeck, Class. Quantum Grav. 7 (1990) 751; Phys. Rev. D50 (1994) 4982.

[5] A. Bellini, M. Ciafaloni, P. Valtancoli, Physics Lett. B 357 (1995) 532; Nucl. Phys. B 462 (1996) 453.

[6] M. Welling, Class. Quantum Grav. 13 (1996) 653; Nucl. Phys. B 515 (1998) 436.

[7] P. Menotti, D. Seminara, Ann. Phys. 279 (2000) 282.

[8] P. Menotti, D. Seminara, Nucl. Phys. (Proc. Suppl.) 88 (2000) 132.

[9] P. G. Zograf, L. A. Tahktajan, Math. USSR Sbornik 60 (1988) 143.

[10] G. 't Hooft, Comm. Math. Phys. 117 (1988) 685.

[11] R. Arnowitt, S. Deser and C.W. Misner, in "Gravitation: an introduction to current research" Edited by L.Witten, John Wiley & Sons New York, London 1962.

[12] S.W. Hawking and C. J. Hunter, Class. Quantum Grav. 13 (1996) 2735; G. Hayward, Phys. Rev. D 47 (1993) 3275.

[13] J. Liouville J. Math. Pures Appl. 18 (1853) 71; H. Poincaré, J. Math. Pures Appl. 5 ser.2 (1898) 157; P. Ginsparg and G. Moore, hep-th/9304011; L. Takhtajan, "Topics in quantum geometry of Riemann surfaces: two dimensional quantum gravity", Como Quantum Groups (1994) 541, hep-th/9409088.

[14] L. A. Takhtajan, Mod. Phys. Lett. A11 (1996) 93.

[15] L. Cantini, P. Menotti and of D. Seminara, *Hamitlonian structure and quantization of 2+1 dimensional gravity coupled to particles*, hep-th-0011070.

[16] R. M. Wald, "General Relativity", The University of Chicago Press, Chicago and London (1984).

[17] V. Moncrief, J. Math. Phys. 30 (1989) 2907; A. Hosoya and K. Nakao, Progr. Theor. Phys. 84 (1990) 739; A. Hosoya and K. Nakao, Class. Quantum Grav. 7 (1990) 163 S. Carlip, Phys. Rev. D42 (1990) 2647; Phys. Rev. D45 (1992) 3584; J. Nelson, T. Regge, Phys. Lett. B273 (1991) 213; M. Welling, Class. Quant. Grav. 14 (1997) 3313; H-J. Matschull and M. Welling, Class. Quantum Grav. 15 (1998) 2981.

[18] M. Bourdeau, S. D. Sorkin, Phys. Rev. D 45 (1992) 687

[19] S. Deser, R. Jackiw, Comm. Math. Phys. 118 (1988) 495.